CALCULUS OF VARIATION

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Find the curve on which an extremum of the functional $v[y(x)]= \int_0^{x_1} \sqrt{\frac{1+y'^2}{y}}dx; y(0)=0$ can be achieved if the second boundary point (x_1, y_1) can move along the circumference of the circle $(x-9)^2+y^2 = 9$

Solution:

$$F=\sqrt{\frac{1+y'^2}{y}}$$

- Since F is independent of x,
- The Eulers Equation is given by,
- $\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right]-\frac{\partial F}{\partial x}=0$
- $\frac{\partial F}{\partial y'}=\frac{y'}{y\sqrt{1+y'}}$

- $\left[\frac{\sqrt{1+y\prime^2}}{y} - \frac{y\prime^2}{y\sqrt{1+y\prime^2}}\right]$=a(constant)
- $\frac{1}{y\sqrt{1+y\prime^2}}$=a
- $y'^2 = \frac{1-y^2a^2}{y^2a^2}$
- $\frac{dy}{dx} = \frac{\sqrt{1+y\prime^2 a^2}}{ay}$
- $dx = \frac{aydy}{\sqrt{1+y\prime^2 a^2}}$

Integrating,

a$\int \frac{ydy}{\sqrt{1+y\prime^2 a^2}}$=x−b

Put, 1−$y^2a^2 = t^2$

ydy=$\frac{-tdt}{a^2}$

- $a\int \frac{-tdt}{a^2t}$ =x−b
- $\frac{-1}{a}$t=x−b
- $\frac{-1}{a}\sqrt{1-a^2y^2}$=x−b
- Squaring,
- $\frac{1}{a^2}$$-y^2$=$(x-b)^2$
- Take, $\frac{1}{a^2}$=k^2
- $(x-b)^2$+y^2=k^2 $\rightarrow$ (1)

The boundary condition y(0)=0

∴b=k

Since the integrand is of the form A(x,y)$\sqrt{1+y'^2}$, the transversality condition reduces to orthogonality condition.

Thus the required extremal will be the arc of the circle belonging to $(x-b)^2+y^2=k^2$ which is orthogonal to $(x-9)^2+y^2=9$.

Since B (x_1, y_1) lies on both circles, we have

$x_1{}^2 - 18x_1 + y_1{}^2 = -72 \rightarrow$ ②

$x_1{}^2 - 2bx_1 + y_1{}^2 = 0 \quad \rightarrow$ ③ ($\because$ b=k)

Solving ② and ③, we get

x_1(b-9)=-36 $\rightarrow$ ④

In view of orthogonality of two circles at (x_1, y_1) ,the tangent to $(x-b)^2+y^2=k^2$ at B passes through (9,0) of the given circle.

The equation of tangent to given circle is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)$+c=0 $\rightarrow$ ⑤

From ③ we get,

g=-b, f=0, c=0.

⑤ $\Rightarrow$ $xx_1+yy_1-b(x+x_1)=0$

At (9,0)

$9x_1-b(9+x_1)=0$

$(9-b)x_1=9b \rightarrow$ ⑥

Solving ④ and ⑥

$x_1 b-9x_1=-36$

$-x_1 b+9x_1=9b$

$\Rightarrow b=4$

Sub b in ④

$x_1(4-9)=-36$

$\Rightarrow x_1=\frac{36}{5}$

The required extremal of the functional is

$(x-4)^2+y^2=16$